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COUPLED ELASTIC AND ELECTROMAGNETIC FIELDS IN A DIATOMIC, DIELECTRIC CONTINUUM

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Abstract—Huang's equations for coupled mechanical and electromagnetic fields in diatomic, ionic crystals are extended to accommodate shorter wave lengths and to include the acoustic branches of the dispersion relations for plane waves.

1. INTRODUCTION

EQUATIONS governing coupled mechanical and electromagnetic fields in diatomic, ionic, optically isotropic crystals were given by Huang [1, 2] in terms of a polarization variable, the relative displacement of the ions and the usual variables of the electromagnetic field. In the present paper, the equations are extended to take into account the separate electronic polarizations and the separate displacements of the two ions, the ionic polarization, the two electronic polarization gradients and the two displacement gradients. Aside from a more detailed representation of the atomic and electronic interactions, there are two main effects of the additional considerations: first, the equations are applicable to shorter wave lengths, owing to the inclusion of the displacement and polarization gradients; second, the acoustic branches are included in the dispersion relations for plane waves, in addition to the optical branches and the electromagnetic branch.

In Section 2, the differential equations of motion are given for a diatomic, ionic crystal with NaCl structure. This is done simply by adding the equations involving the magnetic field to a previous set of equations [3] which was restricted to a quasi-static electric field.

Dispersion relations for longitudinal and transverse waves in the [100] direction are derived in Section 3 and an examination of the long wave limits reveals the presence of the acoustical branches as well as the optical branches and electromagnetic branch included in Huang's results. As previously found by Huang, the longitudinal and transverse optical branches have the same long wave limit when the electromagnetic field is taken into account. It is also found that the long wave limits of the optical and electromagnetic branches are not influenced by the polarization and displacement gradients and the long wave limits of the acoustic branches are independent of electrical and magnetic properties.

In the last section, the identity of the static dielectric constant and the low frequency index of refraction of the electromagnetic wave is verified.

2. EQUATIONS OF MOTION

As in a previous paper restricted to the quasi-static electric field, each of the two interpenetrating continua representing the diatomic ionic crystal has its own displacement $u_i^{(\kappa)}, \kappa = 1, 2$ and electronic polarization (per unit area) $P_i^{(\kappa)}$. The superscripts 1 and 2 identify the continuum representations of the lattices of positive and negative ions, respectively.

The stored energy density of deformation and polarization, W^L, is assumed to be a function of the individual small strains

$$S_{ij}^{(\kappa)} = \tfrac{1}{2}(u_{j,i}^{(\kappa)} + u_{i,j}^{(\kappa)}), \qquad \kappa = 1, 2, \tag{1}$$

the individual polarizations, $P_i^{(\kappa)}$, the individual polarization gradients, $P_{j,i}^{(\kappa)}$, the relative displacement

$$u_i^* = u_i^{(2)} - u_i^{(1)} \tag{2}$$

and the relative rotation

$$\omega_{ij}^* = \tfrac{1}{2}(u_{j,i}^* - u_{i,j}^*). \tag{3}$$

Thus,

$$W^L = W^L(S_{ij}^{(1)}, S_{ij}^{(2)}, P_i^{(1)}, P_i^{(2)}, P_{j,i}^{(1)}, P_{j,i}^{(2)}, u_i^*, \omega_{ij}^*). \tag{4}$$

The total polarization per unit area, P_i, is the sum of the electronic polarizations and the ionic polarization

$$P_i = P_i^{(1)} + P_i^{(2)} + q_* u_i^*, \tag{5}$$

where q_* is the ionic charge per unit volume.

Of the variables in (4), only P_i and u_i^* are accounted for in Huang's [1, 2] equations of motion.

When the Maxwell electric self-field, E_i^{MS}, is quasi-static, it satisfies

$$\varepsilon_{ijk} E_{k,j}^{MS} = 0, \tag{6}$$

where ε_{ijk} is the unit alternating tensor. In that case, the total potential energy density, W, is the sum of W^L and the energy density associated with E_i^{MS}:

$$W = W^L + \tfrac{1}{2}\varepsilon_0 E_i^{MS} E_i^{MS}, \tag{7}$$

where ε_0 is the permittivity of a vacuum. The accompanying kinetic energy density is

$$T = \sum_{\kappa} \tfrac{1}{2}\rho^{(\kappa)} \dot{u}_i^{(\kappa)} \dot{u}_i^{(\kappa)}, \qquad \kappa = 1, 2, \tag{8}$$

where $\rho^{(1)}$ and $\rho^{(2)}$ are the mass densities of the individual continua and the $\dot{u}_i^{(\kappa)}$ are the velocities.

Again for the case of the quasi-static electric field, it was shown previously [3], by means of an extension of Toupin's [4] variational principle for elastic dielectrics, that the field equations, in addition to (6), are

$$T_{ij,i}^{(\kappa)} + (-1)^{\kappa}(T_{ij,i}^* - T_j^* + q_* E_j^{MS}) + f_j^{(\kappa)} + (-1)^{\kappa} q_* E_j^0 = \rho^{(\kappa)} \ddot{u}_j^{(\kappa)}, \tag{9}$$

$$E_{ij,i}^{(\kappa)} + E_j^{(\kappa)} + E_j^{MS} + E_j^0 = 0, \tag{10}$$

$$\varepsilon_0 E_{i,i}^{MS} + P_{i,i}^{(1)} + P_{i,i}^{(2)} + q_* u_{i,i}^* = 0, \tag{11}$$

where

$$T_{ij}^{(\kappa)} = \frac{\partial W^L}{\partial S_{ij}^{(\kappa)}}, \qquad E_i^{(\kappa)} = -\frac{\partial W^L}{\partial P_i^{(\kappa)}}, \qquad E_{ij}^{(\kappa)} = \frac{\partial W^L}{\partial P_{j,i}^{(\kappa)}}, \qquad T_i^* = \frac{\partial W^L}{\partial u_i^*}, \qquad T_{ij}^* = \frac{\partial W^L}{\partial \omega_{ij}^*} \tag{12}$$

and $f_i^{(\kappa)}$ and E_i^0 are the body forces and external electric field, respectively.

The magnetic field is incorporated into these equations simply by changing (6) to

$$\varepsilon_{ijk}E^{MS}_{k,j}+\dot{B}_i = 0 \tag{13}$$

and adding the equations

$$\mu_0^{-1}\varepsilon_{ijk}B_{k,j} = \varepsilon_0\dot{E}^{MS}_i+\dot{P}^{(1)}_i+\dot{P}^{(2)}_i+q_*\dot{u}^*_i, \tag{14}$$

$$B_{i,i} = 0, \tag{15}$$

where B_i is the magnetic flux density and μ_0 is the magnetic permeability of a vacuum (the magnetic susceptibility being neglected). The potential energy density becomes

$$W = W^L+\tfrac{1}{2}\varepsilon_0E^{MS}_iE^{MS}_i+\tfrac{1}{2}\mu_0^{-1}B_iB_i, \tag{16}$$

while the energy density of deformation and polarization (7), the kinetic energy density (8), the field equations (9)–(11) and the constitutive equations (12) remain unchanged.

It was shown previously [3] that, for the NaCl-type structure, the energy density of deformation and polarization is

$$\begin{aligned} W^L = \tfrac{1}{2}\sum_{\kappa,\lambda}&(a^{\kappa\lambda}_{ij}P^{(\kappa)}_iP^{(\lambda)}_j+b^{\kappa\lambda}_{ijkl}P^{(\kappa)}_{j,i}P^{(\lambda)}_{l,k}+c^{\kappa\lambda}_{ijkl}S^{(\kappa)}_{ij}S^{(\lambda)}_{kl}+2d^{\kappa\lambda}_{ijkl}P^{(\kappa)}_{(j,i)}S^{(\lambda)}_{kl}\\ &+\sum_{\kappa}(a^{*\kappa}u^*_iP^{(\kappa)}_i+d^{*\kappa}\omega^*_{ij}P^{(\kappa)}_{[j,i]}+a^{**}u^*_i\,u^*_i+c^{**}\omega^*_{ij}\omega^*_{ij})\\ &+\sum_{\kappa}(b^{\kappa 0}P^{(\kappa)}_{i,i}+c^{\kappa 0}S^{(\kappa)}_{ii}), \end{aligned} \tag{17}$$

where $P^{(\kappa)}_{(j,i)}$ and $P^{(\kappa)}_{[j,i]}$ denote the symmetric and antisymmetric parts of $P^{(\kappa)}_{j,i}$, respectively. Also,

$$\begin{aligned} a^{\kappa\lambda}_{ij} &= a^{\lambda\kappa}_{ij} = a^{\kappa\lambda}_{11}\delta_{ij},\\ b^{\kappa\lambda}_{ijkl} &= b^{\lambda\kappa}_{ijkl} = b^{\kappa\lambda}\delta_{ijkl}+b^{\kappa\lambda}_{12}\delta_{ij}\delta_{kl}+b^{\kappa\lambda}_{44}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})+b^{\kappa\lambda}_{77}(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}),\\ c^{\kappa\lambda}_{ijkl} &= c^{\lambda\kappa}_{ijkl} = c^{\kappa\lambda}\delta_{ijkl}+c^{\kappa\lambda}_{12}\delta_{ij}\delta_{kl}+c^{\kappa\lambda}_{44}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}),\\ d^{\kappa\lambda}_{ijkl} &= d^{\kappa\lambda}\delta_{ijkl}+d^{\kappa\lambda}_{12}\delta_{ij}\delta_{kl}+d^{\kappa\lambda}_{44}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}), \end{aligned} \tag{18}$$

where δ_{ij} (or δ_{ijkl}) is unity when its indices are alike and zero otherwise and

$$b^{\kappa\lambda} = b^{\kappa\lambda}_{11}-b^{\kappa\lambda}_{12}-2b^{\kappa\lambda}_{44},\qquad c^{\kappa\lambda} = c^{\kappa\lambda}_{11}-c^{\kappa\lambda}_{12}-2c^{\kappa\lambda}_{44},\qquad d^{\kappa\lambda} = d^{\kappa\lambda}_{11}-d^{\kappa\lambda}_{12}-2d^{\kappa\lambda}_{44}. \tag{19}$$

It may be observed that material constants with superscripts $\kappa\lambda = 11$ or 22 denote interactions within one of the two component continua whereas constants with superscripts $\kappa\lambda = 12$ or 21 denote interactions between the two continua. Also, the asterisks identify material constants associated with the relative displacement u^*_i.

Upon substituting (17) in (12) and the resulting expressions in the field equations, we find

$$\begin{aligned} &\sum_{\lambda}[c^{\kappa\lambda}\delta_{ijkl}u^{(\lambda)}_{l,ki}+c^{\kappa\lambda}_{12}u^{(\lambda)}_{i,ij}+c^{\kappa\lambda}_{44}(u^{(\lambda)}_{j,ii}+u^{(\lambda)}_{i,ij})]\\ &\quad+\sum_{\lambda}[d^{\lambda\kappa}\delta_{ijkl}P^{(\lambda)}_{l,ki}+d^{\lambda\kappa}_{12}P^{(\lambda)}_{i,ij}+d^{\lambda\kappa}_{44}(P^{(\lambda)}_{j,ii}+P^{(\lambda)}_{i,ij})]\\ &\quad+(-1)^{\kappa}\sum_{\lambda}[d^{*\lambda}(P^{(\lambda)}_{j,ii}-P^{(\lambda)}_{i,ij})-(-1)^{\lambda}c^{**}(u^{(\lambda)}_{j,ii}-u^{(\lambda)}_{i,ij})]\\ &\quad-(-1)^{\kappa}[a^{*1}P^{(1)}_j+a^{*2}P^{(2)}_j+a^{**}(u^{(2)}_j-u^{(1)}_j)-q_*E^{MS}_j-q_*E^0_j]+f^{(\kappa)}_j = \rho^{(\kappa)}\ddot{u}^{(\kappa)}_j, \end{aligned} \tag{20}$$

$$\sum_{\lambda} [d^{\kappa\lambda}\delta_{ijkl}u^{(\lambda)}_{l,ki} + d^{\kappa\lambda}_{12}u^{(\lambda)}_{i,ij} + d^{\kappa\lambda}_{44}(u^{(\lambda)}_{j,ii} + u^{(\lambda)}_{i,ij}) + (-1)^{\lambda}d^{*\kappa}(u^{(\lambda)}_{j,ii} - u^{(\lambda)}_{i,ij})]$$

$$+ \sum_{\lambda} [b^{\kappa\lambda}\delta_{ijkl}P^{(\lambda)}_{l,ki} + b^{\kappa\lambda}_{12}P^{(\lambda)}_{i,ij} + b^{\kappa\lambda}_{44}(P^{(\lambda)}_{j,ii} + P^{(\lambda)}_{i,ij}) + b^{\kappa\lambda}_{77}(P_{j,ii} - P_{i,ij})]$$

$$-a^{1\kappa}_{11}P^{(1)}_{j} - a^{2\kappa}_{11}P^{(2)}_{j} - a^{**}(u^{(2)}_{j} - u^{(1)}_{j}) + E^{MS}_{j} + E^{0}_{j} = 0, \tag{21}$$

$$\varepsilon_0 E^{MS}_{i,i} + P^{(1)}_{i,i} + P^{(2)}_{i,i} + q_*(u^{(2)}_{i,i} - u^{(1)}_{i,i}) = 0, \tag{22}$$

$$\varepsilon_{ijk}E^{MS}_{k,j} + \dot{B}_i = 0, \tag{23}$$

$$\mu_0^{-1}\varepsilon_{ijk}B_{k,j} - \varepsilon_0\dot{E}^{MS}_i - \dot{P}^{(1)}_i - \dot{P}^{(2)}_i - q_*\dot{u}^*_i = 0, \tag{24}$$

$$B_{i,i} = 0. \tag{25}$$

These are the equations of motion for the coupled elastic and electromagnetic fields in the continuum representation of a diatomic ionic crystal with NaCl structure.

3. WAVES IN THE [100] DIRECTION

We consider plane waves in the [100], or x_1, direction:

$$u^{(\kappa)}_i = K^{(\kappa)}\,\mathrm{e}^{i\psi}, \qquad P^{(\kappa)}_i = L^{(\kappa)}_i\,\mathrm{e}^{i\psi}, \qquad E^{MS}_i = M_i\,\mathrm{e}^{i\psi}, \qquad B_i = N_i\,\mathrm{e}^{i\psi}, \tag{26}$$

where $\psi = \xi x_1 - \omega t$ and $K^{(\kappa)}_i$, $L^{(\kappa)}_i$, M_i and N_i are constants.

(a) *Longitudinal waves*

In the case of longitudinal waves,

$$u^{(\kappa)}_2 = u^{(\kappa)}_3 = 0, \qquad P^{(\kappa)}_2 = P^{(\kappa)}_3 = 0, \qquad E^{MS}_2 = E^{MS}_3 = 0, \tag{27}$$

$$u^{(\kappa)}_i = K^{(\kappa)}_1\,\mathrm{e}^{i\psi}, \qquad P^{(\kappa)}_1 = L^{(\kappa)}_1\,\mathrm{e}^{i\psi}, \qquad E^{MS}_1 = M_1\,\mathrm{e}^{i\psi}. \tag{28}$$

Then, from (24), $B_i = 0$; i.e. there is no coupling with the magnetic field, but the Maxwell electric self-field is coupled with the displacement and polarization fields. The solution is, therefore, the same as that found previously [3] for the case of the quasi-static electric field, with dispersion relation

$$\Delta_L = 0, \tag{29}$$

where Δ_L is the determinant with elements

$$\begin{aligned}
\Delta_{11} &= \rho^{(1)}\omega^2 - a^{**} - q_*\varepsilon_0^{-1} - c^{11}_{11}\xi^2, & \Delta_{31} &= \Delta_{13},\\
\Delta_{12} &= a^{**} + q_*^2\varepsilon_0^{-1} - c^{12}_{11}\xi^2, & \Delta_{32} &= \Delta_{23},\\
\Delta_{13} &= a^{*1} + q_*\varepsilon_0^{-1} - d^{11}_{11}\xi^2, & \Delta_{33} &= -a^{11}_{11} - \varepsilon_0^{-1} - b^{11}_{11}\xi^2,\\
\Delta_{14} &= a^{*2} + q_*\varepsilon_0^{-1} - d^{21}_{11}\xi^2, & \Delta_{34} &= -a^{21}_{11} - \varepsilon_0^{-1} - b^{21}_{11}\xi^2,\\
\Delta_{21} &= \Delta_{12}, & \Delta_{41} &= \Delta_{4},\\
\Delta_{22} &= \rho^{(2)}\omega^2 - a^{**} - q_4^2\varepsilon_0^{-1} - c^{11}_{11}\xi^2, & \Delta_{42} &= \Delta_{24},\\
\Delta_{23} &= -a^{*1} - q_*\varepsilon_0^{-1} - d^{12}_{1}\xi^2, & \Delta_{43} &= \Delta_{34},\\
\Delta_{24} &= -a^{*2} - q_*\varepsilon_0^{-1} - d^{22}_{11}\xi^2, & \Delta_{44} &= -a^{22}_{11} - \varepsilon_0^{-1} - b^{22}_{11}\xi^2.
\end{aligned} \tag{30}$$

The long wave limit of the longitudinal optical branch is obtained from

$$\lim_{\xi\to 0}\Delta_L = \omega^2\{\rho^{(1)}\rho^{(2)}\omega^2[(a_{11}^{11}+\varepsilon_0^{-1})(a_{11}^{22}+\varepsilon_0^{-1})-(a_{11}^{12}+\varepsilon_0^{-1})^2]-(\rho^{(1)}+\rho^{(2)})D_L\} \quad (31)$$

where

$$D_L = \begin{vmatrix} a_{11}^{11}+\varepsilon_0^{-1} & a_{11}^{12}+\varepsilon_0^{-1} & a^{*1}+q_*\varepsilon_0^{-1} \\ a_{11}^{21}+\varepsilon_0^{-1} & a_{11}^{22}+\varepsilon_0^{-1} & a^{*2}+q_*\varepsilon_0^{-1} \\ a^{*1}+q_*\varepsilon_0^{-1} & a^{*2}+q_*\varepsilon_0^{-1} & a^{**}+q_*^2\varepsilon_0^{-1} \end{vmatrix}. \quad (32)$$

Hence, the limiting frequency of the longitudinal optical branch is

$$\omega_{LO} = \{D_L/\bar{\rho}[(a_{11}^{11}+\varepsilon_0^{-1})(a_{11}^{22}+\varepsilon_0^{-1})-(a_{11}^{12}+\varepsilon_0^{-1})^2]\}^{\frac{1}{2}}, \quad (33)$$

where $\bar{\rho} = \rho^{(1)}\rho^{(2)}/(\rho^{(1)}+\rho^{(2)})$, i.e. $\bar{\rho}$ is the reduced mass density.

The long wave behavior of the longitudinal acoustic branch of the dispersion relation is obtained from

$$\lim_{\omega,\xi\to 0}\Delta_L = \lim_{\omega,\xi\to 0}[(c_{11}^{11}+c_{11}^{22}+2c_{11}^{12})\xi^2-(\rho^{(1)}+\rho^{(2)})\omega^2]D_L. \quad (34)$$

Hence, at long wave lengths, the frequency of the longitudinal acoustic branch is given by

$$\omega_{LA} = \xi[(c_{11}^{11}+c_{11}^{22}+2c_{11}^{12})/(\rho^{(1)}+\rho^{(2)})]^{\frac{1}{2}}. \quad (35)$$

(b) *Transverse waves*

Of the two similar transverse waves, with displacements in the directions of x_2 and x_3, respectively, we choose the former for examination:

$$u_1^{(\kappa)} = u_3^{(\kappa)} = 0, \qquad P_1^{(\kappa)} = P_3^{(\kappa)} = 0, \qquad E_1^{MS} = E_3^{MS} = 0, \qquad B_1 = B_2 = 0, \quad (36)$$

$$u_2^{(\kappa)} = K_2^{(\kappa)}\,e^{i\psi}, \qquad P_2^{(\kappa)} = L_2^{(\kappa)}\,e^{i\psi}, \qquad E_2^{MS} = M_2\,e^{i\psi}, \qquad B_3 = N_3\,e^{i\psi}. \quad (37)$$

In the case of the quasi-static electric field the absence of (24) permitted E_2^{MS}, as well as B_3, to be zero; but both must be non-zero when the full electromagnetic equations are imposed.

To find a form of the dispersion determinant similar to (30), it is useful first to express M_2 and N_3 in terms of $K_2^{(\kappa)}$ and $L_2^{(\kappa)}$ through the use of (23) and (24):

$$M_2 = -i\omega N_3/\xi = -\varepsilon_\xi^{-1}[L_2^{(1)}+L_2^{(2)}+q_*(K_2^{(2)}-K_2^{(1)})], \quad (38)$$

where

$$\varepsilon_\xi = \varepsilon_0-\xi^2/\mu_0\omega^2. \quad (39)$$

When the results (38), along with (36) and (37), are inserted in the remaining equations of motion, the latter are satisfied if

$$\Delta_T^{EM} = 0, \quad (40)$$

where Δ_T^{EM} is the determinant with elements

$$
\begin{aligned}
\Delta'_{11} &= \rho^{(1)}\omega^2 - a^{**} - q_*\varepsilon_\xi^{-1} - (c^{11}_{44} - c^{**})\xi^2, & \Delta'_{31} &= \Delta'_{13},\\
\Delta'_{12} &= a^{**} + q_*^2\varepsilon_\xi^{-1} - (c^{21}_{44} + c^{**})\xi^2, & \Delta'_{32} &= \Delta'_{23},\\
\Delta'_{13} &= a^{*1} + q_*\varepsilon_\xi^{-1} - (d^{11}_{44} - d^{*1})\xi^2, & \Delta'_{33} &= -a^{11}_{11} - \varepsilon_\xi^{-1} - (b^{11}_{44} + b^{11}_{77})\xi^2,\\
\Delta'_{14} &= a^{*2} + q_*\varepsilon_\xi^{-1} - (d^{21}_{44} - d^{*2})\xi^2, & \Delta'_{34} &= -a^{21}_{11} - \varepsilon_\xi^{-1} - (b^{21}_{44} + b^{21}_{77})\xi^2,\\
\Delta'_{21} &= \Delta'_{12}, & \Delta'_{41} &= \Delta'_{14},\\
\Delta'_{22} &= \rho^{(2)}\omega^2 - a^{**} - q_*\varepsilon_\xi^{-1} - (c^{22}_{44} - c^{**})\xi^2, & \Delta'_{42} &= \Delta'_{24},\\
\Delta'_{23} &= -a^{*1} - q_*\varepsilon_\xi^{-1} - (d^{12}_{44} + d^{*1})\xi^2, & \Delta'_{43} &= \Delta'_{34},\\
\Delta'_{24} &= -a^{*2} - q_*\varepsilon_\xi^{-1} - (d^{22}_{44} + d^{*2})\xi^2, & \Delta'_{44} &= -q^{22}_{11} - \varepsilon_\xi^{-1} - (b^{22}_{44} + b^{22}_{77})\xi^2.
\end{aligned}
\tag{41}
$$

The long wave limit of the transverse optical branch is obtained from

$$\lim_{\xi \to 0} \Delta_T^{EM} = \omega^4\{\rho^{(1)}\rho^{(2)}\omega^{(2)}[(a^{11}_{11} + \varepsilon_0^{-1})(a^{22}_{11} + \varepsilon_0^{-1}) - (a^{12}_{11} + \varepsilon_0^{-1})^2] - (\rho^{(1)} + \rho^{(2)}D_L\}, \tag{42}$$

so that the limiting frequency is

$$\omega^{EM}_{TO} = \{D_L/\bar{\rho}[(a^{11}_{11} + \varepsilon_0^{-1})(a^{22}_{11} + \varepsilon_0^{-1}) - (a^{12}_{11} + \varepsilon_0^{-1})^2]\}^{\frac{1}{2}}, \tag{43}$$

i.e. as found by Huang [1, 2], the same as the limiting frequency of the longitudinal optical branch.

The long wave, low frequency behavior is given by

$$\lim_{\omega,\xi \to 0} \Delta_T^{EM} = \lim_{\omega,\xi \to 0} [(c^{11}_{44} + c^{22}_{44} + 2c^{12}_{44})\xi^2 - (\rho^{(1)} + \rho^{(2)})\omega^2]D_\xi, \tag{44}$$

where

$$D_\xi = (\varepsilon_0\mu_0 D_L\omega^2 - \xi^2 D)/(\varepsilon_0\mu_0\omega^2 - \xi^2) \tag{45}$$

and

$$D = \begin{vmatrix} a^{11}_{11} & a^{12}_{11} & a^{*1} \\ a^{21}_{11} & a^{22}_{11} & a^{*2} \\ a^{*1} & a^{*2} & a^{**} \end{vmatrix}. \tag{46}$$

Thus, there are two branches, with limiting behaviors

$$\omega_{EM} = \xi(D/\varepsilon_0\mu_0 D_L)^{\frac{1}{2}}, \tag{47}$$

$$\omega^{EM}_{TA} = \xi[(c^{11}_{44} + c^{22}_{44} + 2c^{12}_{44})/(\rho^{(1)} + \rho^{(2)})]^{\frac{1}{2}}. \tag{48}$$

The first of these is the long wave end of the electromagnetic branch, as found by Huang [1, 2]; the second is the long wave end of the transverse acoustic branch: identical with that obtained for the case of the quasi-static electric field [3].

It will be observed that the long wave limits of the longitudinal and transverse optical branches, (33) and (43), and the electromagnetic branch (47) are independent of the material constants b, c and d, i.e. independent of the polarization and displacement gradients. Also the long wave limits of the acoustic branches, (35) and (48), depend only on the mass density and the elastic stiffness $c_{11}^{\kappa\lambda}$ and $c_{44}^{\kappa\lambda}$. Hence, as far as the limiting behavior at the long wave end is concerned, the results, here, conform with Huang's, with the addition of the acoustic branches. However, as may be seen from the determinants (30) and (41), when the wave length diminishes from infinity, the elastic part couples with the remainder and the results diverge from Huang's.

4. INDEX OF REFRACTION AND DIELECTRIC CONSTANT

Recalling that the velocity of electromagnetic waves in a vacuum is

$$c = (\varepsilon_0\mu_0)^{-\frac{1}{2}}, \tag{49}$$

we see, from (47), that the low frequency index of refraction of electromagnetic waves in the dielectric is

$$n = c\xi/\omega = (D_L/D)^{\frac{1}{2}}. \tag{50}$$

That the square of the low frequency index of refraction, D_L/D, is equal to the static dielectric constant, defined as

$$K_0 = 1+(P/\varepsilon_0 E^{MS})_{\omega=0}, \tag{51}$$

may be shown, conveniently, by deducing Huang's formula for the dielectric constant from (20) and (21). As shown in [3], Huang's equations of motion may be found from (20) and (21) by discarding all spatial derivative terms (thereby restricting the applicability of the equations to infinite or, at least, very long wave lengths) and expressing the residue in terms of $w_i(=\bar{\rho}^{\frac{1}{2}}u_i^*)$, P_i and E_i^{MS}, with the result (omitting $f_i^{(\kappa)}$ and E_i^0):

$$\ddot{w}_i = b_{11}w_i+b_{12}E_i^{MS}, \tag{52}$$

$$P_i = b_{21}w_i+b_{22}E_i^{MS}, \tag{53}$$

where Huang's constants b_{11}, b_{22}, b_{12}, b_{21} are expressed in terms of constants appearing in the present paper by

$$b_{11} = -D/\bar{\rho}(a_{11}^{11}a_{11}^{22}-a_{11}^{12}a_{11}^{21}), \qquad b_{22} = (a_{11}^{11}+a_{11}^{22}-2a_{11}^{12})/(a_{11}^{11}a_{11}^{22}-a_{11}^{12}a_{11}^{21}), \tag{54}$$

$$b_{12} = b_{21} = [(a_{11}^{11}-a_{11}^{12})a^{*2}+(a_{11}^{22}-a_{11}^{12})a^{*1}-q_*(a_{11}^{11}a_{11}^{22}-a_{11}^{12}a_{11}^{21})/(a_{11}^{11}a_{11}^{22}-a_{11}^{12}a_{11}^{21})\bar{\rho}^{\frac{1}{2}}. \tag{55}$$

Introducing a factor $e^{i\omega t}$ and eliminating w_i between (52) and (53), we have

$$P_i/E_i^{MS} = [b_{22}(b_{11}+\omega^2)-b_{12}^2]/(b_{11}+\omega^2). \tag{56}$$

Inserting this in (51), we find Huang's formula for the static dielectric constant

$$K_0 = 1+(b_{11}b_{22}-b_{12}^2)/\varepsilon_0 b_{11}. \tag{57}$$

Upon substituting the formulas for b_{11}, b_{22} and b_{12}, given in (54) and (55), into (57), we find, after some algebraic manipulations,

$$K_0 = D_L/D = n^2, \tag{58}$$

as required.

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Абстракт—Обобщаются уравнения Хуанга, касающиеся сопряженных механических и электромагнитных полей, в двухатомных, ионных кристаллах, для приспособления более коротких волн и учёта акустических ветвей для зависимостей дисперсии плоских волн.